Resource Allocation in Marketing

by Kalyan Raman

Resource allocation is a central issue in marketing. A broad conceptualization includes allocation of resources over both market entities and across time. Resource allocation over market entities encompasses such problems as the optimal amount to spend on each of the media in the firm’s communications mix and the determination of the optimal levels of the components of the marketing mix. The exemplar for resource allocation over time is the determination of the optimal advertising budget – should it be constant, increase over time, decrease over time or follow a more intricate pattern? I propose a unifying flexible theoretical framework to encompass marketing resource allocation problems and show how prior research fits into it. Finally I develop generalizable and broadly applicable insights from resource allocation problems.

Keywords
Resource allocation, optimization, marginal analysis, dynamic and stochastic optimal

Control, uncertainty, risk, integrated marketing communications, marketing mix, game theory, Kalman filter, Dorfman-Steiner theorem, general resource allocation theorem.

“... marketing is the catalyst for the transmutation of latent resources into actual resources, of desires into accomplishments...”

Peter F. Drucker (1958)

1. Introduction

Inefficient allocation of its resources decreases the profitability of the firm and contributes to societal waste. Among the firm’s resource allocation problems, none is more complex and important than that of allocating marketing resources. Kumar (2008) notes that “the marketing mix allocation problem is the one question that concerns all marketing managers across all industry verticals.” According to Wyner (2008) of Millward Brown Inc., this is “the most critical of marketing decisions,” and, “optimizing the marketing mix can achieve financial improvements of 10% to 20%.” To put this in perspective, note that even if we focus on just a single component of the marketing mix – advertising – the “advertising budget of US companies exceeded $285 billion in 2006 – more than the gross domestic product of Malaysia, Hong Kong or New Zealand” (Gupta/Steenburgh 2008). The unending debates about capitalism versus socialism and everything in between revolve around a resource allocation problem, immense in magnitude and admitting of no easy answer – because the central question is the optimal mode of allocating scarce economic resources. Many important problems in business and society are ultimately resource allocation problems.

With economic growth stagnating, firms struggling to survive in a globally competitive environment, and sustainable growth becoming a high priority in the western nations, resource allocation has become an ever more pressing issue in marketing. The most recent definition of marketing crafted by the AMA in 2004 explicitly states that the central task of Marketing is the maximization of value: “Marketing is an organizational function and a set of processes for creating, communicating and delivering value to customers and for managing customer relationships in ways that benefit the organization and its stakeholders.” (Wilkie/Moore 2007). The centrality of resource allocation flows directly from this mandate to maximize value.
But the resource allocation practices of managers do not typically come even close to value maximization. As Mantrala (2006) notes, “Marketing resource allocation decisions are complicated and managers tend to address them with fairly arbitrary and simplifying heuristics and decision rules.” But blind application of simple heuristics based on arbitrary or ad hoc reasoning will usually generate seriously suboptimal profits (Raman 1990), and may sometimes result in disastrous consequences (Mantrala 1996).

In this article, I present a general framework for optimal resource allocation in marketing. The generality serves two purposes – it makes the framework flexible enough to accommodate most of the marketing resource allocation problems of practical interest, and generates new insights by delineating the scope of applicability of extant resource allocation rules and showing how to modify them to optimize resource allocation. The general resource allocation theorem proved in this paper can improve the profitability of any firm making marketing mix allocation decisions – it is particularly applicable to the consumer packaged goods sector in which third-party vendors make much of the data needed for mix modeling available (Wyner 2008). While we do not delve into the issue in this paper, it is important to note that the statistical problem of making inferences about the market response function parameters is a difficult one, additionally exacerbated by the endogeneity resulting from situations in which the manager may have attempted to partially optimize the allocation of resources.


There is no dearth of resource allocation models in the marketing literature. The models span the spectrum of methodological virtuosity and conceptual richness. The literature review will establish the need for a general framework that incorporates the key issues in optimal resource allocation. The key contribution of this paper is its generalization of the classic Dorfman-Steiner theorem to accommodate both dynamics and uncertainty.

2.1. Literature Review

The earliest and best-known model of resource allocation is the Dorfman-Steiner theorem. Dorfman and Steiner’s (1954) fundamental paper established optimality conditions (necessary but not sufficient) for the marketing mix to maximize the profits of the firm, but their static and deterministic assumptions about sales response limited the applicability of their results to real-world marketing mix decisions. Market response functions are typically dynamic because of – among other reasons – consumer memory (which is the case in advertising), or stocking up items that are aggressively promoted by marketers. Researchers subsequently extended their work to incorporate dynamics (Kalish 1983; Nerlove/Arrow 1962), competition (Lambin/Nuert/Bultez 1975) or uncertainty (Aykac et. al 1989; Jagpal/Brick 1982; Morey/McCann 1983; Nguyen 1985; Tapiero 1975; 1978), but the joint effect of both dynamics and uncertainty on the optimal marketing mix was unaddressed. Raman/Chatterjee (1995) developed dynamically optimal pricing policies under demand uncertainty and Raman (1990) analyzed the dynamic optimality of the ratio rule for advertising under demand uncertainty. Raman (2006) solved a class of boundary value problems in stochastic optimal control of advertising – each boundary value problem corresponding to a specific assumption about the salvage value of sales at the end of a finite horizon. However, the impact of error variance on expected profits under joint optimization of advertising, price and other components of the marketing mix remained unexplored under dynamic demand uncertainty. Little (1966) pointed out that “a company needs a control system for its marketing variables.” The problem is important because demand is a function of the marketing mix, and both dynamic and uncertain effects shape markets. I will first develop a general stochastic and dynamic framework for the marketing mix problem within which the firm’s control problem can be precisely defined. An advantage of this framework is that it permits a generalized version of the classic Dorfman-Steiner (1954) results.

2.2. A General Model of Sales

I propose a model formulation that is general enough to include the majority of the sales response functions that have appeared in the marketing literature. A general model of sales should satisfy the following criteria: (1) it should incorporate the influence of the marketing mix, (2) it should admit dynamic marketing effects, and (3) it should permit stochastic evolution of sales. The first criterion requires a mathematical system with all the marketing mix controls that the manager can manipulate, the second criterion calls for differential equations, and the third for stochastic processes. Consequently, the methodology of controlled stochastic differential equations is the natural candidate for this problem. The control variables are precisely the marketing mix variables – the 4 Ps-available to the firm. These include such variables as price, advertising, promotions, personal selling, distribution and quality. Denoting the cumulative sales process by \(X(t)\), the marketing mix by a vector \(u(t)\) of controls, a general stochastic and dynamic model of sales response to the mix is given by the controlled stochastic differential equation:

\[
dX = f(X, u, t)dt + \sigma(X, u, t)dW.
\] (1)

Equation (1) is analogous to a dynamic discrete-time econometric model of the form,

\[
X_i = f(X_{i-1}, u_{i-1}, t) + \sigma (X_{i-1}, u_{i-1}, t)\epsilon_i
\]

in which \(f\) is the explained part and \(\epsilon_i\) is a disturbance term accounting for unexplained variations in sales in period “\(t\).” In continuous time, the Wiener process incre-
ment \( dW(t) \) plays the role of \( \varepsilon_t \). Equation (1) is a stochastic differential equation (SDE) of the Itô type, and its appropriateness as a mathematical model for our problem is rooted in the following noteworthy features. Continuous-time models are attractive for the following reasons. Continuous-time models generate stochastic differential equations which have a number of distinct advantages over their discrete-time counterparts which are stochastic difference equations. The utilization of SDE technology finds precedence in (Bökér 1987; Chen/Juá 1992; Möller 2000; Raman 1990; Raman/Chatterjee 1995; Raman/Naik 2004; Rao 1986). Second, the SDE allows the marketing mix to potentially influence not only the level of sales but also the shape of the probability distribution of sales, an attractive feature of this technology, because it increases the richness and diversity of market behavior that can be mirrored in this framework. Third, solutions of (1) generate Markov processes (Arnold 1974). The marketing literature contains substantial precedence for the use of Markov processes in modeling market response (Hanssens/Parsons/Schultz 2001). Models of the type shown in equation (1) may be estimated by Kalman Filter methodology, as shown, for example, in Naik/Raman (2003) and Naik/Raman/Winer (2005). The interested reader may consult those papers for an in-depth discussion of the estimation methodology.

There are many empirically validated choices possible for the function \( f(.) \) appearing in equation (1). For example, it has a generalized Koyck model structure in both Naik/Raman (2003) and Raman/Naik (2004) — the authors assume \( f(.) = \beta_0 u + \beta_n y + \kappa u v - (1 - \lambda) x \). The function \( f(.) \) has the form \( f(.) = \beta_0 u - \delta x \) in Raman (2006) and in the classic paper of Nerlove/Arrow (1962).

2.3. A General Framework for Resource Allocation

The resource allocation problem may be stated in full generality in the following terms. Given the general dynamic and stochastic model of sales shown in equation (1), the firm’s problem is to determine \( \{u(t)\} \), where \( u(t) \) is a control vector consisting of the marketing mix variables (\( p(t), a(t), d(t), q(t) \)) where \( p, a, d, \) and \( q \) denote price, advertising, distribution and quality respectively, and \( u(t) \) satisfies:

\[
\begin{align*}
\text{max} & \quad E_x \left\{ \int_0^T e^{rt} \pi(x, u, t) \, dt \right\},
\end{align*}
\]

where \( \pi(x, u, t) \) is the instantaneous profit at time “\( t \),” and \( E_x \) is the expectation conditional upon \( X(0) = X_0 \). The instantaneous profit \( \pi(x, u, t) = (p(t) - C(x, q)) f(x, u, t) - a(t) - q(t) - C_v \), where \( C(x, q) \) is the cost associated with producing a unit at quality level “\( q \)” and \( C_v \) is the fixed cost. The dependence of \( C(x, q) \) upon “\( x \)” incorporates potential experience curve effects that may reduce production cost with the growth of cumulative output. The planning horizon \( T \) may be either infinite or finite — both have been used in the marketing literature. To characterize \( u(t) \), I use the Hamilton-Jacobi-Bellman framework, in which the key player is the value function, defined as

\[
V(x, t) = \max_{\pi(x, u, t)} \left\{ E_x \left[ \int_0^T e^{rt} \pi(x, u, s) \, ds \right] \right\}
\]

called the value function, denotes the maximum expected profit that can be achieved by using the optimal marketing mix over the remaining horizon \([t, T]\), starting from an arbitrary state \( x \).

3. General Resource Allocation Problem

I extend the classic Dorfman-Steiner theorem to encompass dynamic and stochastic market response functions and determine the optimal resource allocation through stochastic optimal control theory. One limitation of my generalization is that it does not include the effect of competition. Of course, neither does the classic Dorfman-Steiner theorem, which has subsequently been extended to account for competitive effects (Lambin/Naert/Bultez 1975); however the Lambin et al. paper does not incorporate dynamics and uncertainty. An extension of the Dorfman-Steiner theorem to the stochastic, dynamic and competitive case would require stochastic differential games which I do not consider here. Naik/Raman/Winer (2005) show how this may be accomplished for an oligopoly consisting of five brands assuming a Lanchester modeling framework to model the dynamic interaction between the players. However, they ignore uncertainty — thus the generalization of the Dorfman-Steiner theorem to the stochastic, dynamic and competitive case remains an open research question. This is not surprising since such a generalization requires stochastic differential games which are very difficult to solve except in the linear-quadratic case in which the state equation describing the evolution of sales is linear and the profit function is quadratic.

Statement of the Classic Dorfman-Steiner Theorem

Stated in the notation of equations (1) and (2), the classic Dorfman-Steiner Theorem states that the necessary conditions for the marketing mix to maximize the firm’s profit in the static and deterministic case are (see Mantrala 2006):

\[
ff_p = -1/\lambda f_q = -1/\lambda f_a = C, \quad ff_q = -1/\lambda f_p = C, \quad ff_a = -1/\lambda f_q = C
\]

It is often useful to state optimal decision rules in terms of elasticities. For example, the optimal monopoly price may be stated as \( p^* = (p_{eq}+(\varepsilon_p+1))-C \), where \( C \) is the production unit cost function and \( \varepsilon_p \) is the price elasticity (Hanssens/Parsons/Schultz 2001). When finding the profit-maximizing combination of price and advertising, it is well-known that the Dorfman-Steiner theorem can also be stated in terms of price and advertising elasticities as follows: \( \varepsilon_{ad}/R = -\varepsilon_p\varepsilon_a \), where \( R \) is the revenue, \( \varepsilon_p \) is the price elasticity, and \( \varepsilon_a \) is the advertising elasticity. Hanssens/Parsons/Schultz (2001) provide a detailed and insightful discussion of these issues in their classic book.
Statement of the General Resource Allocation Theorem

The general resource allocation theorem extends the Dorfman-Steiner theorem to include uncertain and dynamic effects, yielding the following resulting necessary conditions for the marketing mix to maximize the firm’s profit

\[
f f_x + DS(p) = -1/f_a + DS(a) = -1/f_d + DS(d) = C_x/f_x + DS(q)
\]

where DS(ui) is a correction factor associated with marketing mix component i. In the Appendix, I show that the correction factor is:

\[
DS(u_i) = \sigma V_{xx} \sigma_{x_i}^\delta.
\]

The correction factor reflects adjustments for dynamic and stochastic effects missing from the classic Dorfman-Steiner conditions which are static and deterministic. The notation DS has been chosen deliberately – it reminds us that the correction factor incorporates dynamic and stochastic adjustments that are missing in the original Dorfman-Steiner conditions. Furthermore, the structure of the correction factor DS is such that the corrections for dynamics and uncertainty enter multiplicatively, so that if either feature – dynamics or uncertainty – is absent, no correction is required.

As Mantrala (2006) points out, “The D-S theorem does not directly give the optimal values of the marketing policy variables but rather the conditions that will be satisfied when the optimal values are found.” It is precisely those conditions – shown in equation (3) – that become the generalized Dorfman-Steiner conditions displayed in equation (4), and exactly as in the classic deterministic static case, so too the generalized Dorfman-Steiner conditions also do not directly yield the optimal values of the marketing mix variables but rather they state the conditions that will be satisfied when the optimal levels are found after correcting for dynamic and stochastic effects. The correction term DS(u_i) corresponding to marketing mix variable u_i shows the exact magnitude of the correction to the classic Dorfman-Steiner conditions required in order to account for the dynamic and stochastic effects in market response function that the classic Dorfman-Steiner theorem ignore.

3.1. How the Resource Allocation Theorem Yields the Dorfman-Steiner Theorem as a Special Case

A useful first step to understanding the general resource allocation theorem is to satisfy ourselves that it reduces to the classic Dorfman-Steiner theorem under static or deterministic conditions. The general resource allocation theorem reduces to the original Dorfman-Steiner conditions in the following cases. (A) Static Response and Cost Functions. In this case, the correction term DS(u_i) is zero for each of the mix components and equation (4) reduces to equation (3). (B) No uncertainty. Once again, the correction term DS(u_i) is zero for each of the mix components and equation (4) reduces to equation (3). (C) Response uncertainty exists but the error variance is not influenced by the marketing mix. The correction term DS(u_i) is again zero for each of the mix components and equation (4) coincides with equation (3). (D) Response uncertainty exists but each component of the marketing mix has the same effect on error variance. (E) Linearity of the value function in the state variable x.

3.2. Interpretation of Allocation Results

The purpose of this section is to reveal the logic underlying the special cases so that the logic becomes clear in the general case. Cases (A) and (B) are direct consequences of the nature of the correction factor – since that factor accounts for dynamic and stochastic effects, it is natural that no correction is needed when those effects are missing in the response and cost functions. Case (C) is not as transparent as the first two cases because it shows that uncertainty, in and of itself, will not change the optimal allocation of marketing resources unless the marketing mix instruments influence the error variance. While Cases (A) and (B) reassure us that the general resource allocation theorem is a valid extension of the classic Dorfman-Steiner theorem, Case (C) extends the scope of the classic theorem by showing that it holds even in an uncertain and dynamic environment provided that the marketing mix leaves the error variance unaffected. At the same time, Case (C) warns us that the classic Dorfman-Steiner theorem will lead to suboptimal resource allocation when the marketing mix does affect the error variance, and that, in such a case, the general resource allocation theorem developed here becomes imperative. Case (D) is subtler than Case (C) because it states that, even when marketing mix instruments influence the error variance, the optimal allocation of marketing resources will not change unless the marketing mix instruments differentially influence the error variance. In other words, if DS(u_i) is the same for each marketing mix component, then the classic Dorfman-Steiner theorem will suffice to find the optimal allocation of resources. This is clear from equation (4) because, if DS(p) = DS(a) = DS(d) = DS(q), then the correction term simply cancels out and equation (4) reduces to equation (3). However, if, say, price (p) has a different effect on error variance than, say, advertising (a), so that DS(p) ≠ DS(a), then, the classic Dorfman-Steiner theorem will lead to a suboptimal allocation of resources – and the general resource allocation theorem will increase the expected profit by readjusting the allocation of resources to maximize the expected profit. Equation (4) shows exactly how to readjust the resources. Case (E) is the most technical – it states that the correction term is not needed when the value function is linear in the state variable x. Mathematically the reason is obvious – when the value function is linear in the state variable x, the second derivative of the value function with respect to x must vanish, which immediately forces the correction term to zero since it is proportional to the second derivative of the value function. Therefore the relevant substantive question here is –
what causes linearity of the value function? The answer is complicated because many combinations of real-world market response function and cost function factors can cause linearity of the value function. The most frequent combination of market response function and cost function factors leading to a linear value function is linear state dynamics (a popular example being the famous Nerlove-Arrow 1962 model) and quadratic cost function (Naik/Raman 2003). To sum up the general resource allocation theorem in the most general case, we may view this theorem in the following way: it shows how to exploit all the components of market response controllable through the marketing mix – statics, dynamics and variability of response – to maximize expected profit.

### 3.3. Extant Optimal Allocation Results as Special Cases of the Resource Allocation Theorem

Most of the extant work on marketing mix response modeling and optimization can be accommodated within the general framework developed here because the response function \( f(.) \), the variance specification \( \sigma(.) \) and the cost function \( C(.) \) are all completely arbitrary except for the very mild assumption about continuity. The continuity assumption is mild because it places few restrictions on market response, production and cost behavior; dynamics or uncertainty – continuity is assumed merely to exploit the power of calculus and stochastic optimal control in deriving the general resource allocation theorem. The classic Dorfman-Steiner (1954) paper and the large body of subsequent work all assume continuity. The Dorfman-Steiner (1954) paper is the simplest special case within the framework of the general resource allocation theorem – it corresponds to lack of dynamics (static response function and static cost function) and to zero variance in the model specification. The classic Nerlove-Arrow (1962) model extended the Dorfman-Steiner (1954) paper by allowing dynamic response (albeit at the expense of assuming a specific functional form) and it is a special case of the general resource allocation theorem obtained by assuming that the variance is zero and the function \( f(.) \) is linear in goodwill and advertising. The Jagpal/Brick (1982) paper corresponds to the case of a static but stochastic response function which is certainly accommodated within the framework of the general resource allocation theorem. The Kalish (1983) paper corresponds to the case in which the response function is dynamic but not stochastic; moreover the function \( f(.) \) contains only price “\( p \)” as its argument, the cost function \( C(.) \) declines with cumulative output, and the planning horizon \( T \) is infinite. The Raman/Chatterjee (1994) paper corresponds to the case of a dynamic and stochastic response function in which the function \( f(.) \) contains only price “\( p \)” as its argument, the cost function \( C(.) \) declines with cumulative output, and the planning horizon \( T \) is infinite. The general resource allocation theorem yields the conditions for the optimal policies developed in those papers. As another example, the Naik/Raman (2003) and Raman (1990) papers correspond to the case in which \( f(.) \) contains only the marketing communication components as its arguments and \( C(.) \) is a quadratic function, Naik/Raman (2003) assume that the planning horizon \( T \) is infinite, whereas Raman (1990) permits the planning horizon \( T \) to be finite. Finite horizon problems subsume infinite horizon problems because the solution to the infinite horizon problem can be obtained as a limiting case by letting \( T \) tend to infinity. Moreover, as is well-known, finite horizon problems are generally much more difficult than their infinite horizon counterparts (Bass et al. 2005), and the general resource allocation theorem in this paper covers both those cases. The Naik/Raman (2003) paper solves the general problem of optimizing the integrated marketing communications mix, accounting for synergy among the media. Since the Naik/Raman (2003) model is subsumed within the general marketing mix model, it follows that the general resource allocation theorem accommodates budgeting and allocation problems that arise in the context of the integrated marketing communications mix. The Raman (2006) paper solves a finite horizon problem corresponding to the assumption that \( f(.) \) contains only advertising as its argument. Other papers on marketing mix modeling and optimization similarly fit into the general framework of this paper by appropriately defining the response, variance and cost functions.

### 3.4. Applying the Generalized Dorfman-Steiner Theorem

The resource allocation problem is challenging because computation of the correction factors \( DS(u_i) \) requires solving a nonlinear stochastic control problem. We illustrate the technique through the following example based on empirically validated assumptions. Consider the following response function, used by many authors in the marketing literature (see Naik/Raman 2003 for the connection of square-root response to linear response through a variable transformation):

\[
dx = (-\delta x + \beta_1 \sqrt{u_1} + \beta_2 \sqrt{u_2}) \, dt + \sigma u_1 \, dW.\tag{5}
\]

The square-root structure guarantees the empirically well-established phenomenon of diminishing returns. The variables \( u_1(t) \) and \( u_2(t) \) are advertising expenditures on two different media – say print and TV. The instantaneous profit function \( \pi(t) = m \, f(x, u, t) - u_1(t) - u_2(t) \), where \( m \) is the profit margin and \( f(x, u, t) = (-\delta x + \beta_1 \sqrt{u_1} + \beta_2 \sqrt{u_2}) \). The variance of the error term is not constant but fluctuates with \( u_1 \) – the infinitesimal error variance is \( \sigma^2 u_1^2 \). One way that such a variance structure arises for the error term is when the parameter \( \beta_1 \) is a stochastic process varying randomly around a constant level. To apply the generalized Dorfman-Steiner theorem, we need the correction factors \( DS(u_1) \) and \( DS(u_2) \). From the definition \( DS(u_i) = \sigma \, \frac{\partial f_{ui}}{\partial\sigma} \), we immediately conclude that \( DS(u_i) = 0 \) because the derivative of the infinitesimal error variance with respect to \( u_i \) is zero since the infinitesimal error variance is a function of \( u_i \) but not of \( u_2 \). To calculate \( DS(u_i) \), we need the value...
function $V(x)$ which we find by solving the Hamilton-Jacobi-Bellman (HJB) equation for this problem:

$$-V_r - \frac{4mx(\sigma V_{x} - 2V + \mu(\sigma V_{x} - 2\beta_j^2))}{\sigma V_{x} - 2} = 0$$

(6)

In equation (6), $V_r = \frac{\partial V}{\partial x}$, $V_{xx} = \frac{\partial^2 V}{\partial x^2}$, and $V_t = \frac{\partial V}{\partial t}$. We sketch the solution of the nonlinear partial differential equation (6) in Appendix B. Here we display the optimality results.

Value Function for the Stochastic and Dynamic Model

$$V(x) = \frac{(\beta_j^2 + \beta_j^1 + \sigma_1^2 + 2\sigma_1^2 \delta - \sqrt{\beta_j^2 + \beta_j^1 + \sigma_1^2 + 2\sigma_1^2 \delta} - 4\beta_j^1(\tau + 2\delta)\sigma^2)}{2\beta_j^1 \sigma^2}$$

(7)

Percentage Gain Over the Dynamic Model Ignoring Uncertainty

Percentage Gain = $A/B$, where

$$A = 1000x_1 + \delta$$

$$B = m(4\gamma^2 \delta + \frac{2\gamma^2 \delta}{\gamma(r + \delta)} + \frac{4\gamma^2 \delta}{\gamma + \delta})$$

(8)

$$u_2(x) = \frac{1}{2\beta_j^2} \frac{\sqrt{\beta_j^2 + \beta_j^1 + \sigma_1^2 + 2\sigma_1^2 \delta - \sqrt{\beta_j^2 + \beta_j^1 + \sigma_1^2 + 2\sigma_1^2 \delta} - 4\beta_j^1(\tau + 2\delta)\sigma^2}}{\beta_j^1 \sigma^2}$$

(13)

The expressions for the optimal allocations are messy – Fig. 2 shows $u_1$ as a function of sales (x) and Fig. 3 shows $u_2$ as a function of sales. Both policies decrease as sales increases.

Ratio of $u_2(x)$ to $u_1(x)$

$$\frac{u_2(x)}{u_1(x)}$$

(14)

To check that the above allocation ratio makes sense, note that it reduces to the deterministic allocation ratio when there is no uncertainty. For $\sigma = 0$, Ratio$_{u_2, u_1} = 1$.

Correction Factors for Optimal Resource Allocation

$$DS(u_1) = \frac{\beta_j^2 + \beta_j^1 + \sigma_1^2 + 2\sigma_1^2 \delta - \sqrt{\beta_j^2 + \beta_j^1 + \sigma_1^2 + 2\sigma_1^2 \delta} - 4\beta_j^1(\tau + 2\delta)\sigma^2}{\beta_j^2 \sigma^2}$$

(10)

$$DS(u_2) = 0$$

(11)
The behavior of the optimal ratio as a function of $\sigma$ is particularly interesting. Recall that the parameter $\sigma$ measures the uncertainty about the effectiveness $\beta_1$ of the first medium. As the level of uncertainty $\sigma$ about the first medium increases relative to the other medium, it is deployed less and the second medium is deployed more heavily, as shown in Fig. 4.

### 3.5 Modeling Market Response and Uncertainty - The Payoff

In their classic book on market response models, Hanssens et al. (2001) establish that the information return on investment for market response analysis is high, and go on to note that “Companies have (...) achieved superior performance because they used market response. Nothing attracts the attention of management more than a concept that works – and market response does.” The general resource allocation theorem shows that the power of market response analysis can be significantly improved by judiciously harnessing the twin forces of market response dynamics and uncertainty. Opportunities to increase expected profit go unheeded when resource allocations are made without accounting for the dynamic effects and uncertainty that are the rule rather than the exception in marketing phenomena. To roughly assess the magnitude of the opportunities, consider the results of reallocation of marketing mix resources at Samsung Electronics. Shankar (2008) reports the following results realized by Samsung, attributable at least in part to marketing resource reallocation – its brand value increased by 30%, its annual sales increased by 25%, and its net income grew by $0.8 billion. And these spectacular results flowed from simply reallocating – not optimizing! Application of the general resource allocation theorem to guide the reallocation of resources could have enabled Samsung to realize at least these gains, and potentially, even greater returns because the theorem shows how to judiciously exploit market response dynamics and uncertainty to maximize profitability. Fig. 1 dramatically makes the point.

Researchers have long recognized the fundamental relevance of market response dynamics, and, while they have acknowledged that uncertainty plays an important role, researchers have frequently ignored the effect of uncertainty in resource allocation decisions. This is because researchers often assume that uncertainty is only relevant for decision makers who are not risk-neutral. Some researchers model the effect of uncertainty through a utility function that generally reflects risk aversion. When the variance is not a function of the marketing mix and the decision maker is risk neutral, researchers have generally downplayed the role of uncertainty in determining the optimal marketing program. The general resource allocation theorem shows that, even under risk-neutrality, ignoring uncertainty is conceptually incorrect and moreover, will generate suboptimal profits. Application of the conditions developed in the general resource allocation theorem generates closed-loop policies that take optimal advantage of dynamics and uncertainty to maximize profitability.

While there are different ways of classifying policies, the distinction between open-loop and closed-loop policies is a fundamental one, in terms of both structure and performance (Erickson 1991; Tapiero 1978). The strategies offering greatest value to managers in uncertain and risky environments are closed-loop (or feedback) policies, because they adapt optimally to changing market conditions. Closed-loop policies facilitate optimal utilization of uncertainty to improve expected profit but open-loop policies cannot influence the impact of uncertainty on expected profit. Open-loop policies are predetermined as a function of time for the entire planning horizon and cannot be revised later. A closed-loop policy is a function of the state, and is like a contingent plan (Jorgensen 1981). Such a policy utilizes all information available on an ongoing basis. The two are equivalent as long as there are no unpredictable inputs to the system (Starr/Ho 1969). Faced with uncertainty, closed-loop policies will do better, on the average, than open-loop ones (Dreyfuss 1965).

Researchers have typically conducted detailed analyses of uncertainty in models containing risk aversion, estimation uncertainty or multiplicative error variance. This paper shows that additive uncertainty and risk-neutrality also have important implications for decision makers formulating dynamically optimal marketing strategy. By using closed-loop policies, the decision-maker can exploit uncertainty in beneficial ways, and the analysis of uncertainty remains important when the decision-maker is risk-neutral, estimation uncertainty is negligible and the error structure is additive. The general resource allocation theorem proved in this paper shows the conditions that hold when all the marketing mix policies are at their optimal levels. By applying the general resource allocation theorem to the specific response, cost and variance specifications that prevail in any particular case, the decision maker can find the closed-loop policies that will dynamically maximize the expected profit.

Mantrala (2006) notes that “Important advances in knowledge about the optimal allocation of marketing resources could have enabled Samsung to realize at least these gains, and potentially, even greater returns because the theorem shows how to judiciously exploit market response dynamics and uncertainty to maximize profitability. Fig. 1 dramatically makes the point.
resources have typically occurred as a result of advances in response measurement and/or optimization models and methods.” The resource allocation technology developed in this paper illustrates Mantrala’s (2006) comment – the general resource allocation theorem is an outcome of the increasing sophistication of response model measurement and optimization made available by the methodologies of stochastic differential equations and stochastic optimal control. Dynamically optimal policies crafted with stochastic control techniques take maximal advantage of uncertainty to improve the firm’s performance. Hence, the methods of stochastic optimal control have central significance for the marketing mix optimization problem. They furnish the natural framework to capture the dynamics and uncertainties of market response, and provide a powerful methodology to extend the Dorfman-Steiner conditions for optimizing the marketing mix. Closed-loop policies take maximal advantage of changing market conditions and yield managerially feasible and implementable ways of exploiting uncertainty to maximize expected profit.

4. Conclusions

Hanssens et al. (2001) state the case for market response function analysis beautifully in the following words: “... for every brand and product category, there exists a process generating its sales. We have seen that market response makes this process known, and because it is known, manageable. In a world of intense competition, what better thing is there for a company to know and to be able to do?” The methodology developed in this paper is a natural outgrowth of the Hanssens et al. (2001) line of reasoning – not only market response but also the variability of response is manageable. To manage both dynamically, new resource allocation rules are needed. And that need is the raison d’être of the general resource allocation theorem. By dynamically managing both market response and variability of response, expected profitability can be significantly improved, as shown explicitly in the context of integrated marketing communications in Raman/Naik (2004), and by the practical experience of firms undertaking a careful reallocation of resources, such as Samsung Electronics and many other business firms (Kumar 2008; Shankar 2008; Wyner 2008).

Researchers have typically focused on uncertainty in models containing risk aversion, estimation uncertainty or multiplicative error variance. This paper shows that additive uncertainty and risk-neutrality should not be ignored. Closed-loop policies optimally exploit uncertainty and by doing so, dramatically increase profitability.

The general resource allocation theorem uses the methodologies of stochastic differential equations and stochastic control techniques – the natural framework to capture the dynamics and uncertainties of market response, and a powerful analytical foundation to extend the Dorfman-Steiner conditions for optimizing the marketing mix. Dynamically optimal policies crafted with these methodologies take maximal advantage of uncertainty to optimize the firm’s performance. Closed-loop policies take maximal advantage of changing market conditions and yield managerially feasible and implementable ways of exploiting uncertainty to maximize expected profit. Today’s markets are dynamic and evolve in highly unpredictable ways, making these techniques increasingly relevant to managers.

Appendix A

The value function \( V(x, t) \) satisfies the Hamilton-Jacobi-Bellman equation (Fleming/Rishel 1975), where \( V_t = \partial V/\partial x, V_{xx} = \partial^2 V/\partial x^2, \) and \( V_t = \partial V/\partial t: \)

\[
V_t + \max_x \left[ e^{-r(x)}(p(x, t) + V_f(x, t) + V_{xx} \sigma^2(x, t))/2 \right] = 0
\]

In this problem, \( \pi(t) = (p(t) – C(x, q)) f(x, u, t) – a(t) – q(t) – C_p \) where \( C(x, q) \) is the cost associated with producing a unit at quality level “x,” and \( C_p \) is the fixed cost. Differentiate the maximand in Bellman’s equation with respect to \( u \) and equal the gradient vector to zero.

\[
e^{-r(x)} \pi_x + V_{xx} \sigma^2 = 0
\]

This yields the system of equations below.

First Order Condition for \( p: \)

\[
e^{-r(x)} \left[ (p – C) f_x + \sigma \sigma_x \right] + V_x f_x + \sigma \sigma_x V_{xx} = 0
\]

First Order Condition for \( q: \)

\[
e^{-r(x)} \left[ (p – C) f_q + \sigma \sigma_q \right] + V_x f_q + \sigma \sigma_q V_{xx} = 0
\]

First Order Condition for \( x: \)

\[
e^{-r(x)} \left[ (p – C) f_x + \sigma \sigma_x \right] + V_x f_x + \sigma \sigma_x V_{xx} = 0
\]

First Order Condition for \( a: \)

\[
e^{-r(x)} \left[ (p – C) f_a + \sigma \sigma_a \right] + V_x f_a + \sigma \sigma_a V_{xx} = 0
\]

Assuming that \( f_x \neq 0 \), divide the First Order Condition for mix element \( u_x \) by \( f_x \), and define the term \( \text{DS}(u_x) = \frac{\sigma V_{xx}}{f_x} \). The term \( \text{DS}(u_x) \) reflects the effects of response dynamics and uncertainty. Then the equation for each mix element can be made equal to \( –(V_x + e^{-r(x)} (p – c)). \) This immediately yields the generalized Dorfman-Steiner conditions. Note that the Dorfman-Steiner conditions are only necessary, not sufficient for optimality. As in the classic Dorfman-Steiner theorem, here too it is assumed that the second-order conditions for sufficiency are satisfied.

Appendix B

The HJB equation for our problem is the following nonlinear partial differential equation:

\[
-\nabla V + \frac{4m \left( \sigma^2 V_{xx} – 2 \right) x + \left( \sigma^2 V_x – 2 \right) x^2 – 4 \left( \sigma^2 V_{xx} – 2 \sigma V_{xx} – 2 \right) x \right)}{4 \sigma^2 V_x} = 0
\]

Collect all the terms into a single fraction:

\[
-\nabla V + \frac{4m \left( \sigma^2 V_{xx} – 2 \right) x + \left( \sigma^2 V_x – 2 \right) x^2 – 4 \left( \sigma^2 V_{xx} – 2 \right) x \right)}{4 \sigma^2 V_x} = 0
\]

In order to satisfy the above equation, the numerator of this fraction must equal zero. This results in a new nonlinear partial differential equation (PDE):

\[
-\nabla V + \frac{4m \left( \sigma^2 V_{xx} – 2 \right) x + \left( \sigma^2 V_x – 2 \right) x^2 – 4 \left( \sigma^2 V_{xx} – 2 \right) x \right)}{4 \sigma^2 V_x} = 0
\]

This is still too hard to solve by known techniques. But assuming that the planning horizon is very long (infinite horizon), the time derivative \( V \) becomes \( V \) and the result-
ing equation is a nonlinear ordinary differential equation (ODE). Using the method of undetermined parameters, we conjecture a polynomial (in x) solution to the above ODE. After trying different orders of the polynomial, we find that a quadratic does the job. Substituting the quadratic conjecture into the ODE yields a set of algebraic equations for the unknown coefficients of the quadratic. By simultaneously solving them, we find the solution to the ODE. Simultaneously solving these equations results in three sets of solutions – one of them yields the value function that corresponds to ignoring uncertainty, and the other two solutions dominate this value function. The other two functions jointly define the optimal value function – each is optimal in a different domain of values of x. The separating point between the domains can be explicitly computed. The solutions and figures shown in the paper correspond to the domain where the third set of solutions define the optimal value function. The algebraic derivations are cumbersome and have been implemented in Mathematica. Interested readers may obtain the rest of the details of the optimization algorithm from the author.

Note

The figures are produced by choosing values of the parameters that are consistent with the range of estimated values that my co-authors and I have obtained in our prior research, resulting in the following choice of parameter values: $\beta_1 = 1.03$, $\beta_2 = 1$, $\delta = 0.05$, $\sigma = 0.8$, $m = 1$ and $r = 0.10$.

References
